

# $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes: generator matrices and duality <sup>\*</sup>

J. Borges; C. Fernández; J. Pujol; J. Rifà; M. Villanueva <sup>†</sup>

## Abstract

A code  $\mathcal{C}$  is  $\mathbb{Z}_2\mathbb{Z}_4$ -additive if the set of coordinates can be partitioned into two subsets  $X$  and  $Y$  such that the punctured code of  $\mathcal{C}$  by deleting the coordinates outside  $X$  (respectively,  $Y$ ) is a binary linear code (respectively, a quaternary linear code). In this paper  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes are studied. Their corresponding binary images, via the Gray map, are  $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes, which seem to be a very distinguished class of binary group codes.

As for binary and quaternary linear codes, for these codes the fundamental parameters are found and standard forms for generator and parity check matrices are given. For this, the appropriate inner product is deduced and the concept of duality for  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes is defined. Moreover, the parameters of the dual codes are computed. Finally, some conditions for self-duality of  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes are given.

## 1 Introduction

Let  $\mathbb{Z}_2$  and  $\mathbb{Z}_4$  be the ring of integers modulo 2 and 4 respectively. Let  $\mathbb{Z}_2^n$  denote the set of all binary vectors of length  $n$  and let  $\mathbb{Z}_4^n$  be the set of all quaternary vectors of length  $n$ . Any non-empty subset  $\mathcal{C}$  of  $\mathbb{Z}_2^n$  is a binary code and a subgroup of  $\mathbb{Z}_2^n$  is called a *binary linear code* or a  $\mathbb{Z}_2$ -linear code. Equivalently, any non-empty subset  $\mathcal{C}$  of  $\mathbb{Z}_4^n$  is a quaternary code and a subgroup of  $\mathbb{Z}_4^n$  is called a *quaternary linear code*.

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<sup>†</sup>The authors are members of the Department of Information and Communications Engineering, Universitat Autònoma de Barcelona, 08193-Bellaterra, Spain.

Quaternary linear codes can be viewed as binary codes under the Gray map defined as  $\phi(0) = (0, 0)$ ,  $\phi(1) = (0, 1)$ ,  $\phi(2) = (1, 1)$ ,  $\phi(3) = (1, 0)$ . If  $\mathcal{C}$  is a quaternary linear code, then the binary code  $C = \phi(\mathcal{C})$  (coordinatewise extended) is said to be a  $\mathbb{Z}_4$ -linear code. The notions of dual code of a quaternary linear code  $\mathcal{C}$ , denoted by  $\mathcal{C}^\perp$ , self-orthogonal code (when  $\mathcal{C} \subseteq \mathcal{C}^\perp$ ) and self-dual code (when  $\mathcal{C} = \mathcal{C}^\perp$ ) are defined in the standard way (see [22]) in terms of the usual inner product for quaternary vectors (see [15]). Since in general the binary code  $C = \phi(\mathcal{C})$  is not linear, it need not have a dual. However, the binary code  $C_\perp = \phi(\mathcal{C}^\perp)$  is called the  $\mathbb{Z}_4$ -dual of  $C = \phi(\mathcal{C})$ .

Since 1994, quaternary linear codes have become significant due to its relationship to some classical well-known binary codes as the Nordstrom-Robinson, Kerdock, Preparata, Goethals or Reed-Muller codes (see [15]). It was proved that the Kerdock code and the Preparata-like code are  $\mathbb{Z}_4$ -linear codes and, moreover, the  $\mathbb{Z}_4$ -dual code of the Kerdock code is the Preparata-like code. Lately, also some families of quaternary linear codes, called *QRM* and *ZRM*, related to the Reed-Muller codes have been studied in [4] and [5], respectively.

Additive codes were first defined by Delsarte in 1973 in terms of association schemes (see [13], [14]). In general, an additive code, in a translation association scheme, is defined as a subgroup of the underlying abelian group. In the special case of a binary Hamming scheme, that is when the underlying abelian group is of order  $2^n$ , the only structures for the abelian group are those of the form  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ , with  $\alpha + 2\beta = n$ . Therefore, the subgroups  $\mathcal{C}$  of  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$  are the only additive codes in a binary Hamming scheme. In order to distinguish them from additive codes over finite fields (see [1], [2], [3], [18]), from now on we will call them  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes.

The binary image of a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code under the extended Gray map defined in Section 2 is called  $\mathbb{Z}_2\mathbb{Z}_4$ -linear code. In [11] and [24], binary perfect 1-error correcting codes (or 1-perfect codes) which are  $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes are described and all such 1-perfect codes are characterized. More examples, such as extended 1-perfect and Hadamard codes which are  $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes are studied in subsequent papers (see [9], [19], [23], [26]). Some notorious codes, e.g. Kerdock-like and Preparata-like codes, can have a  $\mathbb{Z}_4$ -linear structure (see [15]), but they cannot have a  $\mathbb{Z}_2\mathbb{Z}_4$ -linear structure with non-empty binary part (see [10]).

As we have seen, the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes belong to the more general family of additive codes. However, notice that one could think of other families of codes with an algebraic structure that also include the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes; such as mixed group codes and translation invariant propelinear codes.

*Mixed group codes* are defined as subgroups of a group of type  $G = G_1 \times$

$\cdots \times G_r$ , where  $G_i$  is a finite abelian group for all  $i = 1, \dots, r$  (see [16], [21]). Since any finite abelian group has a factorization in cyclic groups, we can also think about a mixed group code as a subgroup of  $\mathbb{Z}_{i_1} \times \cdots \times \mathbb{Z}_{i_s}$ , where the indices  $i_1, \dots, i_s$  are not necessarily different. If we are interested in a binary version of these codes, we need a one-to-one mapping  $\phi$  from  $\mathbb{Z}_{i_j}$  to  $\mathbb{Z}_2^m$  (where  $2^m \geq i_j$ ) for all  $j = 1, \dots, s$ . In [8], it was shown that the indices  $i_1, \dots, i_s$  must be all even, if we want to use a Gray map  $\phi$  which has the classical property that  $d(\phi(i), \phi(i+1)) = 1$ , where  $d(\cdot, \cdot)$  is the Hamming distance between binary vectors. This Gray map is unique, up to coordinate permutation, if the binary image is also Hamming compatible. Moreover, it was also proved that if the binary image of a subgroup of  $\mathbb{Z}_{i_1} \times \cdots \times \mathbb{Z}_{i_s}$ , using such Gray map, is a 1-perfect code, then  $i_j \in \{2, 4\}$ , for all  $j = 1, \dots, s$  (i.e. it is also a  $\mathbb{Z}_2\mathbb{Z}_4$ -linear code).

*Translation invariant propelinear codes* were first defined in 1997 (see [24], [25]). In [24], it was also proved that all such binary codes are group-isomorphic to subgroups of  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \times Q_8^k$ , where  $Q_8$  is the non-abelian quaternion group on eight elements. Hence, abelian translation invariant propelinear codes are exactly all the  $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes.

Most of the concepts on  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes described in this paper have been implemented by the authors as a new package in MAGMA (see [7]). MAGMA is a software package designed to solve computationally hard problems in algebra, number theory, geometry and combinatorics. Currently it supports the basic facilities for linear codes over integer residue rings and Galois rings; moreover, it also supports functions for additive codes over a finite field, which are a generalization of the linear codes over a finite field (see [12, Chapter 119, 120]). However, it does not include functions to work with  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes. For this reason, a beta version of this new package for  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes and the manual with the description of all functions can be downloaded from the web page <http://www.ccg.uab.cat>. For any comment or further information about this package, you can send an e-mail to [support-ccg@deic.uab.cat](mailto:support-ccg@deic.uab.cat).

The aim of this paper is a general study of  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes and the corresponding  $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes. It is organized as follows. In Section 2, we give the definition of  $\mathbb{Z}_2\mathbb{Z}_4$ -additive and  $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes, we find which are the fundamental parameters, and we also discuss about the automorphism groups of these codes. In Section 3, we deduce a standard form for generator matrices of  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes. Section 4 is devoted to study duality for  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes defining the appropriate inner product. In Section 5, we show how the generator and parity check matrices are related and we also compute the parameters of the dual code. Finally, in Section 6, we give some

conditions for self-duality.

## 2 Definitions

From now on, we will focus on  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes  $\mathcal{C}$ , which are subgroups of  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ . We will take an extension of the usual Gray map:  $\Phi : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^n$ , where  $n = \alpha + 2\beta$ , given by

$$\begin{aligned}\Phi(x, y) &= (x, \phi(y_1), \dots, \phi(y_\beta)) \\ \forall x \in \mathbb{Z}_2^\alpha, \forall y &= (y_1, \dots, y_\beta) \in \mathbb{Z}_4^\beta;\end{aligned}$$

where  $\phi : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2^2$  is the usual Gray map, that is,

$$\phi(0) = (0, 0), \phi(1) = (0, 1), \phi(2) = (1, 1), \phi(3) = (1, 0).$$

This Gray map is an isometry which transforms Lee distances defined in the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes  $\mathcal{C}$  over  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$  to Hamming distances defined in the binary codes  $C = \Phi(\mathcal{C})$ . Note that the length of the binary code  $C$  is  $n = \alpha + 2\beta$ .

Since  $\mathcal{C}$  is a subgroup of  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ , it is also isomorphic to an abelian structure like  $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$ . Therefore,  $\mathcal{C}$  is of type  $2^\gamma 4^\delta$  as a group, it has  $|\mathcal{C}| = 2^{\gamma+2\delta}$  codewords and the number of order two codewords in  $\mathcal{C}$  is  $2^{\gamma+\delta}$ .

Let  $X$  (respectively  $Y$ ) be the set of  $\mathbb{Z}_2$  (respectively  $\mathbb{Z}_4$ ) coordinate positions, so  $|X| = \alpha$  and  $|Y| = \beta$ . Unless otherwise stated, the set  $X$  corresponds to the first  $\alpha$  coordinates and  $Y$  corresponds to the last  $\beta$  coordinates. Call  $\mathcal{C}_X$  (respectively  $\mathcal{C}_Y$ ) the punctured code of  $\mathcal{C}$  by deleting the coordinates outside  $X$  (respectively  $Y$ ). Let  $\mathcal{C}_b$  be the subcode of  $\mathcal{C}$  which contains all order two codewords and let  $\kappa$  be the dimension of  $(\mathcal{C}_b)_X$ , which is a binary linear code. For the case  $\alpha = 0$ , we will write  $\kappa = 0$ .

Considering all these parameters, we will say that  $\mathcal{C}$  (or equivalently  $C = \Phi(\mathcal{C})$ ) is of type  $(\alpha, \beta; \gamma, \delta; \kappa)$ . Notice that  $\mathcal{C}_Y$  is a quaternary linear code of type  $(0, \beta; \gamma_Y, \delta; 0)$ , where  $0 \leq \gamma_Y \leq \gamma$ , and  $\mathcal{C}_X$  is a binary linear code of type  $(\alpha, 0; \gamma_X, 0; \gamma_X)$ , where  $\kappa \leq \gamma_X \leq \gamma$ .

**Definition 1** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code, that is a subgroup of  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ . We say that the binary image  $C = \Phi(\mathcal{C})$  is a  $\mathbb{Z}_2\mathbb{Z}_4$ -linear code of length  $n = \alpha + 2\beta$  and type  $(\alpha, \beta; \gamma, \delta; \kappa)$ , where  $\gamma, \delta$  and  $\kappa$  are defined as above.*

Note that  $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes are a generalization of binary linear codes and  $\mathbb{Z}_4$ -linear codes. When  $\beta = 0$ , the binary code  $C = \mathcal{C}$  corresponds to a binary linear code. On the other hand, when  $\alpha = 0$ , the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code

$\mathcal{C}$  is a quaternary linear code and its corresponding binary code  $C = \Phi(\mathcal{C})$  is a  $\mathbb{Z}_4$ -linear code.

Two  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes  $\mathcal{C}_1$  and  $\mathcal{C}_2$  both of type  $(\alpha, \beta; \gamma, \delta; \kappa)$  are said to be *monomially equivalent*, if one can be obtained from the other by permutating the coordinates and (if necessary) changing the signs of certain  $\mathbb{Z}_4$  coordinates. Two  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes are said to be *permutation equivalent* if they differ only by a permutation of coordinates. The *monomial automorphism group* of a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code  $\mathcal{C}$ , denoted by  $MAut(\mathcal{C})$ , is the group generated by all permutations and sign-changes of the  $\mathbb{Z}_4$  coordinates that preserve the set of codewords of  $\mathcal{C}$ , while the *permutation automorphism group* of  $\mathcal{C}$ , denoted by  $PAut(\mathcal{C})$ , is the group generated by all permutations that preserve the set of codewords of  $\mathcal{C}$  (see [17]).

If two  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are monomially equivalent, then, after the Gray map, the corresponding  $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes  $C_1 = \Phi(\mathcal{C}_1)$  and  $C_2 = \Phi(\mathcal{C}_2)$  are isomorphic as binary codes. Note that the inverse statement is not always true.

### 3 Generator matrices of $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes

Let  $\mathcal{C}$  be a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code. Although  $\mathcal{C}$  is not a free module, every codeword is uniquely expressible in the form

$$\sum_{i=1}^{\gamma} \lambda_i u_i + \sum_{j=\gamma+1}^{\gamma+\delta} \mu_j v_j,$$

where  $\lambda_i \in \mathbb{Z}_2$  for  $1 \leq i \leq \gamma$ ,  $\mu_j \in \mathbb{Z}_4$  for  $\gamma + 1 \leq j \leq \gamma + \delta$  and  $u_i, v_j$  are vectors in  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$  of order two and order four, respectively. The vectors  $u_i, v_j$  give us a generator matrix  $\mathcal{G}$  of size  $(\gamma + \delta) \times (\alpha + \beta)$  for the code  $\mathcal{C}$ . Moreover, we can write  $\mathcal{G}$  as

$$\mathcal{G} = \left( \begin{array}{c|c} B_1 & 2B_3 \\ \hline B_2 & Q \end{array} \right), \quad (1)$$

where  $B_1, B_2, B_3$  are matrices over  $\mathbb{Z}_2$  of size  $\gamma \times \alpha$ ,  $\delta \times \alpha$  and  $\gamma \times \beta$ , respectively; and  $Q$  is a matrix over  $\mathbb{Z}_4$  of size  $\delta \times \beta$  with quaternary row vectors of order four.

In [15], it was shown that any quaternary linear code of type  $(0, \beta; \gamma, \delta; 0)$  is permutation equivalent to a quaternary linear code with a generator matrix of the form

$$\mathcal{G}_S = \left( \begin{array}{c|c|c} 2T & 2I_\gamma & \mathbf{0} \\ \hline S & R & I_\delta \end{array} \right), \quad (2)$$

where  $R, T$  are matrices over  $\mathbb{Z}_2$  of size  $\delta \times \gamma$  and  $\gamma \times (\beta - \gamma - \delta)$ , respectively; and  $S$  is a matrix over  $\mathbb{Z}_4$  of size  $\delta \times (\beta - \gamma - \delta)$ . In this section, we will generalize this result for  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes, so we will give a canonical generator matrix for these codes (see [6]).

First, notice that changing ones by twos in the coordinates over  $\mathbb{Z}_2$ , we can see the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes as quaternary linear codes. Let  $\chi$  be the map from  $\mathbb{Z}_2$  to  $\mathbb{Z}_4$ , which is the usual inclusion from the additive structure in  $\mathbb{Z}_2$  to  $\mathbb{Z}_4$ :  $\chi(0) = 0, \chi(1) = 2$ . This map can be extended to the map  $(\chi, Id) : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta$ , which will also be denoted by  $\chi$ .

**Theorem 1** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$ . Then,  $\mathcal{C}$  is permutation equivalent to a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code with canonical generator matrix of the form*

$$\mathcal{G}_S = \left( \begin{array}{cc|cc} I_\kappa & T_b & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \hline \mathbf{0} & S_b & S_q & R & I_\delta \end{array} \right), \quad (3)$$

where  $T_b, T_1, T_2, R, S_b$  are matrices over  $\mathbb{Z}_2$  and  $S_q$  is a matrix over  $\mathbb{Z}_4$ .

*Proof:* Since  $\kappa$  is the dimension of the matrix  $B_1$  over  $\mathbb{Z}_2$  given in (1), the code  $\mathcal{C}$  has a generator matrix of the form

$$\left( \begin{array}{cc|c} I_\kappa & \bar{B}_1 & 2\bar{B}_3 \\ \mathbf{0} & \mathbf{0} & 2\bar{B}_4 \\ \hline \mathbf{0} & \bar{B}_2 & \bar{Q} \end{array} \right),$$

where  $\bar{B}_1, \bar{B}_2, \bar{B}_3$  and  $\bar{B}_4$  are matrices over  $\mathbb{Z}_2$  of size  $\kappa \times (\alpha - \kappa)$ ,  $\delta \times (\alpha - \kappa)$ ,  $\kappa \times \beta$  and  $(\gamma - \kappa) \times \beta$ , respectively; and  $\bar{Q}$  is a matrix over  $\mathbb{Z}_4$  of size  $\delta \times \beta$ .

The quaternary linear code  $\mathcal{C}^-$  of type  $(0, \alpha - \kappa + \beta; \gamma - \kappa, \delta; 0)$  generated by the matrix

$$\left( \begin{array}{cc} \mathbf{0} & 2\bar{B}_4 \\ \hline 2\bar{B}_2 & \bar{Q} \end{array} \right)$$

is permutation equivalent to a quaternary linear code with generator matrix of the form

$$\mathcal{G}^- = \left( \begin{array}{cccc} \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \hline 2S_b & S_q & R & I_\delta \end{array} \right),$$

where the permutation of coordinates fixes the first  $\alpha - \kappa$  coordinates (see [15] or (2)). So, the quaternary linear code  $\chi(\mathcal{C})$  generated by the matrix

$$\left( \begin{array}{ccc} 2I_\kappa & 2\bar{B}_1 & 2\bar{B}_3 \\ \mathbf{0} & \mathbf{0} & 2\bar{B}_4 \\ \hline \mathbf{0} & 2\bar{B}_2 & \bar{Q} \end{array} \right)$$

is permutation equivalent to a quaternary linear code with generator matrix of the form

$$\mathcal{G}_\chi = \left( \begin{array}{cc|cc|c} 2I_\kappa & 2T_b & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \hline \mathbf{0} & 2S_b & S_q & R & I_\delta \end{array} \right).$$

Finally,  $\mathcal{C}$  is permutation equivalent to a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code with generator matrix  $\chi^{-1}(\mathcal{G}_\chi) = \mathcal{G}_S$ .  $\triangle$

**Example 1** Let  $\mathcal{C}_1$  denote the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of type  $(1, 3; 1, 2; 1)$  with generator matrix

$$\mathcal{G} = \left( \begin{array}{c|ccc} 1 & 2 & 2 & 2 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 3 \end{array} \right).$$

The code  $\mathcal{C}_1$  can also be generated by the matrix

$$\left( \begin{array}{c|ccc} 1 & 2 & 2 & 2 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 3 \end{array} \right).$$

The quaternary linear code  $\mathcal{C}^-$  generated by  $\left( \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 3 \end{array} \right)$  is permutation equivalent to a quaternary linear code with generator matrix  $\mathcal{G}^- = \left( \begin{array}{ccc} 1 & 1 & 0 \\ 3 & 0 & 1 \end{array} \right)$ . So, the quaternary linear code  $\chi(\mathcal{C})$  generated by

$$\left( \begin{array}{cccc} 2 & 2 & 2 & 2 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 3 \end{array} \right)$$

is permutation equivalent to a quaternary linear code with generator matrix

$$\mathcal{G}_\chi = \left( \begin{array}{cccc} 2 & 2 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array} \right).$$

Therefore, the code  $\mathcal{C}_1$  is permutation equivalent to a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code with canonical generator matrix

$$\mathcal{G}_S = \chi^{-1}(\mathcal{G}_\chi) = \left( \begin{array}{c|ccc} 1 & 2 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array} \right).$$

**Example 2** Let  $\mathcal{C}_2$  be a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of type  $(3, 4; 3, 1; 3)$  with generator matrix

$$\left( \begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right).$$

By Theorem 1,  $\mathcal{C}_2$  is permutation equivalent to a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code with canonical generator matrix

$$\left( \begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right).$$

## 4 Duality of $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes

For linear codes over finite fields or finite rings there exists the well-known concept of duality. In this section, we will study this concept for  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes. First, we will show that the inner product of elements of a finite abelian group can be uniquely defined. Then, considering the finite group  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$  we will define the notions of duality, as the additive dual code and the  $\mathbb{Z}_2\mathbb{Z}_4$ -dual code, for  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes and its corresponding  $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes, respectively.

Given a finite abelian group  $(G, +)$  of exponent  $m \geq 1$  (so, for each  $b \in G$  we have  $mb = 0$ ), we call *dual group* of  $G$ , denoted by  $\widehat{G}$ , the group of homomorphisms from  $G$  into  $\mathbb{Z}_m$ ,  $\widehat{G} = \text{Hom}(G, \mathbb{Z}_m)$ .

**Example 3** Let  $(G, +)$  be the cyclic group of order four with generator  $a$ , so  $G = \{a, 2a, 3a, 4a = 0\}$ . If we know  $\varphi(a)$ , where  $\varphi \in \widehat{G}$ , we will know the image of any element in  $G$ , because for any  $b \in G$  we have  $b = ia$  and  $\varphi(b) = \varphi(ia) = i\varphi(a) \in \mathbb{Z}_4$ . So, there are four different homomorphisms that we can define over  $G$ :

	$\varphi(0)$	$\varphi(a)$	$\varphi(2a)$	$\varphi(3a)$
$\varphi_0$	0	0	0	0
$\varphi_1$	0	1	2	3
$\varphi_2$	0	2	0	2
$\varphi_3$	0	3	2	1

Note that, in general, for any cyclic group  $(G, +)$  of order  $m$  with generator  $a$ , we can construct all the possible homomorphisms as  $\varphi_k(ia) = ki \in \mathbb{Z}_m$ . Note also that  $\varphi_k(b) + \varphi_s(b) = \varphi_{k+s}(b)$ , for any  $b \in G$ .



It is well-known that  $(\widehat{G}, \cdot)$  is an abelian group by using the operation  $(\varphi \cdot \lambda)(g) = \varphi(g) + \lambda(g)$ , where  $\varphi, \lambda \in \widehat{G}$  and  $g \in G$  (see [20]). The group  $\widehat{G}$  has the same cardinality as  $G$  and both groups are isomorphic (see [20]), but there is no a canonical (or natural) isomorphism from  $G$  to  $\widehat{G}$ .

Assume  $G$  is a cyclic group of order  $m$  and fix a generator  $a \in G$ . Any homomorphism  $\varphi \in \widehat{G} = \text{Hom}(G, \mathbb{Z}_m)$  is defined knowing  $\varphi(a)$ . If  $\varphi(a) = k \in \mathbb{Z}_m$ , this homomorphism will be denoted by  $\varphi_k$  and, for any element  $b = ia \in G$ ,  $\varphi_k(b) = \varphi_k(ia) = i\varphi_k(a) = ik \in \mathbb{Z}_m$ . So, we can define an isomorphism  $G \longrightarrow \widehat{G}$ , such that for all  $c \in G$  we have  $c = ja \mapsto \varphi_j$ . Note that this isomorphism depends on the fixed generator  $a \in G$ .

Let  $G_1, G_2$  be two abelian groups of exponent  $m \geq 1$ . A *bilinear map* of  $G_1 \times G_2$  into  $\mathbb{Z}_m$  is a map

$$\begin{aligned} G_1 \times G_2 &\longrightarrow \mathbb{Z}_m \\ (x_1, x_2) &\longmapsto \langle x_1, x_2 \rangle \end{aligned}$$

such that for  $x_1 \in G_1$  the function  $x_2 \mapsto \langle x_1, x_2 \rangle$  and for  $x_2 \in G_2$  the function  $x_1 \mapsto \langle x_1, x_2 \rangle$  are homomorphisms.

Let  $G$  be an abelian group of exponent  $m \geq 1$ . A special case of bilinear map is

$$G \times \widehat{G} \longrightarrow \mathbb{Z}_m, \quad (4)$$

where  $(b, \varphi) \mapsto \varphi(b) \in \mathbb{Z}_m$ , for all  $b \in G$  and  $\varphi \in \widehat{G}$ . Another special case of bilinear map is the so called *inner product* in  $G$  given by

$$G \times G \longrightarrow \mathbb{Z}_m, \quad (5)$$

where  $(b, c) \mapsto \varphi(b) \in \mathbb{Z}_m$ ,  $\widehat{\varphi} : G \longrightarrow \widehat{G}$  is a fixed isomorphism and  $\varphi = \widehat{\varphi}(c)$ , for all  $b, c \in G$ .

Note that although the bilinear map given by (4) is canonically defined, the inner product defined by (5) depends on the particular isomorphism  $\widehat{\varphi}$  from  $G$  to  $\widehat{G}$  that we use.

Assume again  $G$  is a cyclic group of order  $m$  and fix a generator  $a \in G$ . The inner product in  $G$  is defined uniquely by

$$(b, c) \mapsto \langle b, c \rangle = \varphi_j(b) = \varphi_j(ia) = ji \in \mathbb{Z}_m, \quad (6)$$

where  $b = ia \in G$ ,  $c = ja \in G$  and  $\varphi_j \in \widehat{G} = \text{Hom}(G, \mathbb{Z}_m)$ .

Let  $G'$  be a subgroup of  $G$  generated by an element  $a' \in G'$  of order  $t$ , where  $t \mid m$ . The dual group of  $G'$  could be considered as  $\text{Hom}(G', \mathbb{Z}_t)$  or

$\text{Hom}(G', \mathbb{Z}_m)$  depending on whether the exponent of  $G'$  is  $t$  or  $m$ , respectively. In both cases a generator  $a'$  in  $G'$  is sent to an element of order  $t$  in  $\mathbb{Z}_t$  or  $\mathbb{Z}_m$ , respectively. This situation can be represented by

$$\begin{array}{ccccc} G' & \longrightarrow & \mathbb{Z}_t & \longrightarrow & \mathbb{Z}_m \\ a' & \longmapsto & 1 & \longmapsto & \bar{s} \end{array}$$

where  $\bar{s} \in \mathbb{Z}_m$  is an element of order  $t$ . Then, after fixing a generator  $a' \in G'$  and an element  $\bar{s} \in \mathbb{Z}_m$  of order  $t$ , the inner product of elements of  $G'$  seen as elements in  $G$  is defined uniquely by

$$(b, c) \mapsto \langle b, c \rangle_m = \bar{s} \langle b, c \rangle_t = \bar{s} \varphi_j(b) = \bar{s} \varphi_j(ia') = \bar{s} ji \in \mathbb{Z}_m,$$

where  $b = ia'$ ,  $c = ja'$  and  $\varphi_j \in \text{Hom}(G', \mathbb{Z}_t)$ .

It is well-known (see [20]) that if  $G$  is a finite abelian group, expressed as a product  $G = G_1 \times G_2$ , then  $\widehat{G}$  is isomorphic to  $\widehat{G_1} \times \widehat{G_2}$  under the mapping  $\widehat{G_1} \times \widehat{G_2} \longrightarrow \widehat{G}$ , where the element  $(\lambda_1, \lambda_2) \in \widehat{G_1} \times \widehat{G_2}$  is transformed into an element in  $\widehat{G}$  such that for all  $(x_1, x_2) \in G$

$$(\lambda_1, \lambda_2)(x_1, x_2) = \lambda_1(x_1) + \lambda_2(x_2).$$

Moreover, any finite abelian group  $G$  is isomorphic to a product of cyclic groups, specifically, we can write

$$G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k},$$

where  $1 < m_1 | m_2 | \cdots | m_k$ . Therefore, we have that the inner product, defined by (6) in a cyclic group, can be extended to any finite abelian group in the following way:

**Proposition 1** *Let  $(G, +)$  be a finite abelian group of exponent  $m$  and consider the decomposition*

$$G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k},$$

*where  $1 < m_1 | m_2 | \cdots | m_k = m$  and  $m = s_i m_i$ , for all  $i = 1, \dots, k$ . After fixing a generator  $a_i \in \mathbb{Z}_{m_i}$  in each component and elements  $\bar{s}_i \in \mathbb{Z}_m$  of order  $m_i$ , any element  $u = u_1 a_1 + u_2 a_2 + \cdots + u_k a_k$  in  $G$  is expressed as  $u = (u_1, u_2, \dots, u_k) \in G$  in this fixed generators system.*

*The inner product of elements  $u = (u_1, u_2, \dots, u_k), v = (v_1, v_2, \dots, v_k) \in G$  is defined uniquely by*

$$\langle u, v \rangle_m = \sum_i \bar{s}_i \langle u_i, v_i \rangle_{m_i} = \sum_i \bar{s}_i \varphi_{v_i}(u_i) = \sum_i \bar{s}_i v_i u_i \in \mathbb{Z}_m. \quad (7)$$

Now, consider the finite abelian group  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$  whose elements are vectors of  $\alpha + \beta$  coordinates (the first  $\alpha$  over  $\mathbb{Z}_2$  and the last  $\beta$  over  $\mathbb{Z}_4$ ). By Proposition 1, fixing generators  $a_i = 1 \in \mathbb{Z}_2$ , for  $1 \leq i \leq \alpha$ , and  $a_i \in \{1, 3\} \in \mathbb{Z}_4$ , for  $\alpha + 1 \leq i \leq \alpha + \beta$ , and also fixing the values  $\bar{s}_i = 2$ , for  $1 \leq i \leq \alpha$ , which is the only possible value of order two in  $\mathbb{Z}_4$ , and  $\bar{s}_i = 1 \in \{1, 3\} \subset \mathbb{Z}_4$ , for  $\alpha + 1 \leq i \leq \alpha + \beta$ , we can write the inner product given by (7) in the following way that we will call *standard inner product*:

$$\langle u, v \rangle = 2\left(\sum_{i=1}^{\alpha} u_i v_i\right) + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j \in \mathbb{Z}_4,$$

where  $u, v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ .

Note that although  $\bar{s}_i$  is uniquely defined for  $1 \leq i \leq \alpha$ , the value of  $\bar{s}_i$ , for  $\alpha + 1 \leq i \leq \alpha + \beta$ , can be chosen from  $\{1, 3\}$  and so, we can produce several different presentations for the inner product. Also note that all of these different presentations of the inner product can be reduced to the standard one, as long as in the computation of  $\langle u, v \rangle$  we take the representation of vector  $u$  using the given generators  $a_i$  and the representation of vector  $v$  using the generators  $a'_i = a_i \in \mathbb{Z}_2$ , for  $1 \leq i \leq \alpha$ , and  $a'_i = \bar{s}_i a_i \in \mathbb{Z}_4$ , for  $\alpha + 1 \leq i \leq \alpha + \beta$ .

We can also write the standard inner product as

$$\langle u, v \rangle = u \cdot J_n \cdot v^t,$$

where  $J_n = \left( \begin{array}{c|c} 2I_\alpha & \mathbf{0} \\ \hline \mathbf{0} & I_\beta \end{array} \right)$  is a diagonal matrix over  $\mathbb{Z}_4$ . Note that when  $\alpha = 0$  the inner product is the usual one for  $\mathbb{Z}_4$ -vectors (i.e. vectors over  $\mathbb{Z}_4$ ) and when  $\beta = 0$  it is twice the usual one for  $\mathbb{Z}_2$ -vectors.

Let  $\mathcal{C}$  be a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$  and let  $C = \Phi(\mathcal{C})$  be the corresponding  $\mathbb{Z}_2\mathbb{Z}_4$ -linear code. The *additive orthogonal code* of  $\mathcal{C}$ , denoted by  $\mathcal{C}^\perp$ , is defined in the standard way

$$\mathcal{C}^\perp = \{v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \mid \langle u, v \rangle = 0 \text{ for all } u \in \mathcal{C}\}.$$

We will also call  $\mathcal{C}^\perp$  the *additive dual code* of  $\mathcal{C}$ . The corresponding binary code  $\Phi(\mathcal{C}^\perp)$  is denoted by  $C_\perp$  and called  $\mathbb{Z}_2\mathbb{Z}_4$ -*dual code* of  $C$ . In the case that  $\alpha = 0$ , so when  $\mathcal{C}$  is a quaternary linear code,  $\mathcal{C}^\perp$  is also called the *quaternary dual code* of  $\mathcal{C}$  and  $C_\perp$  the  $\mathbb{Z}_4$ -*dual code* of  $C$ .

The additive dual code  $\mathcal{C}^\perp$  is also a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code, that is a subgroup of  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ . Its weight enumerator polynomial is related to the weight enumerator polynomial of  $\mathcal{C}$  by McWilliams Identity (see [13]). Notice that

$C$  and  $C_\perp$  are not dual in the binary linear sense but the weight enumerator polynomial of  $C_\perp$  is the McWilliams transform of the weight enumerator polynomial of  $C$  (see [13], [24]).

**Lemma 1** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$  and  $\mathcal{C}^\perp$  its additive dual code. Then,  $|\mathcal{C}||\mathcal{C}^\perp| = 2^n$ , where  $n = \alpha + 2\beta$ .*

*Proof:* From the McWilliams Identity,

$$W_{\mathcal{C}^\perp}(X, Y) = \frac{1}{|\mathcal{C}|} W_{\mathcal{C}}(X + Y, X - Y).$$

Taking  $X = Y$  we obtain,

$$|\mathcal{C}^\perp|X^n = \frac{1}{|\mathcal{C}|}(2X)^{n-wt(\mathbf{0})}$$

and hence  $|\mathcal{C}^\perp||\mathcal{C}| = 2^n$ .  $\triangle$

Finally, notice again that one could think on  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes (or  $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes) only as quaternary linear codes (or  $\mathbb{Z}_4$ -linear codes), changing ones by twos in the coordinates over  $\mathbb{Z}_2$ . However, they are not equivalent to the quaternary linear code, since the inner product defined in  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$  gives us that the dual code of a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code is not equivalent to the dual code of the corresponding quaternary linear code. Take, for example,  $\alpha = \beta = 1$  and the vectors  $v = (1, 3)$  and  $w = (1, 2)$ . It is easy to check that  $\langle v, w \rangle = 0$ , so  $v$  and  $w$  are orthogonal. If we change the ones by twos in the coordinates over  $\mathbb{Z}_2$  of these vectors we get  $v' = (2, 3)$  and  $w' = (2, 2)$ , which are not orthogonal in the quaternary sense.

## 5 Parity-check matrices of $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes

In this section, first we will prove two different methods to construct the additive dual code of a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code and we will compute the type of this additive dual code. Then, we will apply one of these two methods to show how to construct a parity-check matrix of a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code, or equivalently a generator matrix of its additive dual code, when the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code is generated by a canonical generator matrix as in (3).

Let  $\mathcal{C}$  be a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$ . Since  $\mathcal{C}$  is a subgroup of  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ , the code  $\mathcal{C}$  could be seen as the kernel of a group homomorphism onto  $\mathbb{Z}_2^{\bar{\gamma}} \times \mathbb{Z}_4^{\bar{\delta}}$ , that is,  $\mathcal{C} = \ker \vartheta$ , where

$$\vartheta : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^{\bar{\gamma}} \times \mathbb{Z}_4^{\bar{\delta}}.$$

The additive dual code  $\mathcal{C}^\perp$  is also the kernel of another group homomorphism onto  $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$ , that is,  $\mathcal{C}^\perp = \ker \bar{\vartheta}$ , where

$$\bar{\vartheta} : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta.$$

The homomorphism  $\vartheta$  can be represented by a matrix  $\mathcal{H}$ , which can be viewed as a parity-check matrix for the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code  $\mathcal{C}$  or as a generator matrix for its additive dual code  $\mathcal{C}^\perp$ . Vice versa, the homomorphism  $\bar{\vartheta}$  can be represented by a matrix  $\mathcal{G}$ , which can be viewed as a parity-check matrix for the additive dual code  $\mathcal{C}^\perp$  or as a generator matrix for the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code  $\mathcal{C}$ .

**Example 4** *The code  $\mathcal{C}_1$  (or the corresponding  $C_1 = \Phi(\mathcal{C}_1)$ ) in Example 1 is a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code (or a  $\mathbb{Z}_2\mathbb{Z}_4$ -linear code) of type  $(1, 3; 1, 2; 1)$  with generator matrix*

$$\mathcal{G}_1 = \left( \begin{array}{c|ccc} 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 3 \end{array} \right).$$

*The generator matrix  $\mathcal{G}_1$  for  $\mathcal{C}_1$  can be also viewed as a parity-check matrix for its additive dual code  $\mathcal{C}_1^\perp$ . Notice also that  $|\mathcal{C}_1| = |C_1| = 2 \cdot 4^2 = 32$ , so by Lemma 1,  $|\mathcal{C}_1^\perp| = 2^7/32 = 4$ .*

In order to construct the additive dual code of a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code, we will need the following maps:  $\xi$  from  $\mathbb{Z}_4$  to  $\mathbb{Z}_2$  which is the usual one modulo two, that is  $\xi(0) = 0$ ,  $\xi(1) = 1$ ,  $\xi(2) = 0$ ,  $\xi(3) = 1$ ; and the identity map  $\iota$  from  $\mathbb{Z}_2$  to  $\mathbb{Z}_4$ , that is  $\iota(0) = 0$ ,  $\iota(1) = 1$ . These maps can be extended to the maps  $(\xi, Id) : \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$  and  $(\iota, Id) : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta$ , which will also be denoted by  $\xi$  and  $\iota$ , respectively. Recall also the map  $\chi$  from  $\mathbb{Z}_2$  to  $\mathbb{Z}_4$  which is the normal inclusion from the additive structure in  $\mathbb{Z}_2$  to  $\mathbb{Z}_4$ , that is  $\chi(0) = 0$ ,  $\chi(1) = 2$ ; and its extension  $(\chi, Id) : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta$ , denoted also by  $\chi$ . We denote by  $\langle \cdot, \cdot \rangle_4$  the standard inner product for quaternary vectors.

**Lemma 2** *If  $u \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ ,  $v \in \mathbb{Z}_4^{\alpha+\beta}$ , then  $\langle \chi(u), v \rangle_4 = \langle u, \xi(v) \rangle$ .*

*Proof:*  $\langle \chi(u), v \rangle_4 = \sum_{i=1}^\alpha (2u_i)v_i + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j = \sum_{i=1}^\alpha (2u_i)(v_i \bmod 2) + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j = \langle u, \xi(v) \rangle. \quad \triangle$

**Corollary 1** *If  $u, v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ , then  $\langle \chi(u), \iota(v) \rangle_4 = \langle u, v \rangle$ .*

*Proof:* By Lemma 2,  $\langle \chi(u), \iota(v) \rangle_4 = \langle u, \xi(\iota(v)) \rangle = \langle u, v \rangle$ .  $\triangle$

**Proposition 2** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$ . Then,*

$$\mathcal{C}^\perp = \xi(\chi(\mathcal{C})^\perp).$$

*Proof:* We know that if  $v \in \mathcal{C}^\perp$ , then  $\langle u, v \rangle = 0$ , for all  $u \in \mathcal{C}$ . By Corollary 1,  $\langle u, v \rangle = \langle \chi(u), \iota(v) \rangle_4 = 0$ . Therefore,  $\xi(\iota(v)) = v \in \xi(\chi(\mathcal{C})^\perp)$  and  $\mathcal{C}^\perp \subseteq \xi(\chi(\mathcal{C})^\perp)$ . On the other hand, if  $v \in \chi(\mathcal{C})^\perp$ , then  $\langle \chi(u), v \rangle_4 = 0$ , for all  $u \in \mathcal{C}$ . By Lemma 2,  $\langle \chi(u), v \rangle_4 = \langle u, \xi(v) \rangle = 0$ . Thus,  $\xi(\chi(\mathcal{C})^\perp) \subseteq \mathcal{C}^\perp$  and we obtain the equality.  $\triangle$

**Proposition 3** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$ . Then,*

$$\mathcal{C}^\perp = \chi^{-1}(\xi^{-1}(\mathcal{C})^\perp).$$

*Proof:* Let  $\mathcal{G}$  be a generator matrix of the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code  $\mathcal{C}$  written as in (1). Then, the quaternary linear code  $\xi^{-1}(\mathcal{C})$  has a generator matrix of the form

$$\begin{pmatrix} 2I_\alpha & \mathbf{0} \\ B_1 & 2B_3 \\ B_2 & Q \end{pmatrix}. \quad (8)$$

We will show that  $v \in \mathcal{C}^\perp$  if and only if  $\chi(v) \in \xi^{-1}(\mathcal{C})^\perp$ . In fact, for each row vector  $f$  in the matrix  $(2I_\alpha \ \mathbf{0})$ , we have  $\langle \chi(v), f \rangle_4 = \sum_{i=1}^\alpha f_i 2v_i = 0$  because there is only one index  $i$  such that  $f_i = 2$ . Moreover, by Corollary 1,  $0 = \langle v, u \rangle = \langle \chi(v), \iota(u) \rangle_4$ , for all  $u \in \mathcal{C}$ .  $\triangle$

The following question we will settle is the computation of the type of the additive dual code of a given  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code  $\mathcal{C}$ . First, we will remember this well-known result for quaternary linear codes, that is for  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes with  $\alpha = 0$ . Then, we will generalize it for  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes, not necessarily quaternary linear codes.

**Lemma 3** [15] *If  $\mathcal{C}$  is a quaternary linear code of type  $(0, \beta; \gamma, \delta; 0)$ , then the quaternary dual code  $\mathcal{C}^\perp$  is of type  $(0, \beta; \gamma, \beta - \gamma - \delta; 0)$ .*

**Theorem 2** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$ . The additive dual code  $\mathcal{C}^\perp$  is then of type  $(\alpha, \beta; \bar{\gamma}, \bar{\delta}; \bar{\kappa})$ , where*

$$\begin{aligned} \bar{\gamma} &= \alpha + \gamma - 2\kappa, \\ \bar{\delta} &= \beta - \gamma - \delta + \kappa, \\ \bar{\kappa} &= \alpha - \kappa. \end{aligned}$$

*Proof:* Let  $\mathcal{G}$  be a generator matrix of the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code  $\mathcal{C}$  written as in (1). Then, the matrix (8) is a generator matrix for the quaternary linear code  $\xi^{-1}(\mathcal{C})$ , which is of type  $(0, \alpha + \beta; \gamma', \delta'; 0)$ , where  $\gamma' = \alpha + \gamma - 2\kappa$  and  $\delta' = \delta + \kappa$ . The value of  $\delta'$  comes from the fact that the  $\kappa$  independent binary vectors of  $(\mathcal{C}_b)_X$  are in  $A$  and, so, the number of independent quaternary vectors of order four becomes  $\delta + \kappa$ . The value of  $\gamma'$  comes from the fact that the cardinality of the quaternary linear code  $\xi^{-1}(\mathcal{C})$  is  $2^{\gamma' + 2\delta'} = 2^{\gamma + 2\delta + \alpha}$ .

By Lemma 3, the quaternary dual code  $\xi^{-1}(\mathcal{C})^\perp$  is of type  $(0, \alpha + \beta; \bar{\gamma}, \bar{\delta}; 0)$ , where  $\bar{\gamma} = \gamma'$  and  $\bar{\delta} = \alpha + \beta - \gamma' - \delta' = \alpha + \beta - (\gamma + \alpha - 2\kappa) - (\delta + \kappa) = \beta - \gamma - \delta + \kappa$ .

Note that the  $\bar{\delta}$  independent vectors in  $\xi^{-1}(\mathcal{C})^\perp$ , restricted to the first  $\alpha$  coordinates, are vectors of order two, because in  $\xi^{-1}(\mathcal{C})$  there are the row vectors of the matrix  $(2I_\alpha \ \mathbf{0})$ . Finally, applying  $\chi^{-1}$  we obtain the additive dual code of  $\mathcal{C}$ . For this additive dual code  $\mathcal{C}^\perp$ , the value of  $\bar{\kappa}$  can be easily computed from the fact that, again, the additive dual coincides with  $\mathcal{C}$ .  $\triangle$

There are two different methods to obtain the additive dual code  $\mathcal{C}^\perp$ , one given by Proposition 2 and another one by Proposition 3. Using any of these two methods, we can construct a generator matrix of  $\mathcal{C}^\perp$ , or equivalently a parity-check matrix of  $\mathcal{C}$ , starting from a generator matrix of  $\mathcal{C}$ . In Example 5, we consider the canonical generator matrix of a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code and apply these two methods to obtain a generator matrix of its additive dual code. Note that the process to obtain this matrix is different using both methods but, in this case, the generator matrices obtained coincide.

Theorem 3 shows how to construct the parity-check matrix of a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code generated by a canonical generator matrix as in (3). This result is proved using the method given by Proposition 2. Notice also that we can apply any of the two methods to any generator matrix, not necessary a canonical generator matrix, to get a parity-check matrix.

**Lemma 4** [15] *If  $\mathcal{C}$  is a quaternary linear code of type  $(0, \beta; \gamma, \delta; 0)$  with canonical generator matrix (2), then the generator matrix of  $\mathcal{C}^\perp$  is*

$$\mathcal{H}_S = \left( \left| \begin{array}{ccc} \mathbf{0} & 2I_\gamma & 2R^t \\ I_{\beta-\gamma-\delta} & T^t & -(S+RT)^t \end{array} \right. \right), \quad (9)$$

where  $R, T$  are matrices over  $\mathbb{Z}_2$  of size  $\delta \times \gamma$  and  $\gamma \times (\beta - \gamma - \delta)$ , respectively; and  $S$  is a matrix over  $\mathbb{Z}_4$  of size  $\delta \times (\beta - \gamma - \delta)$ .

**Theorem 3** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$  with canonical generator matrix (3). Then, the generator matrix of  $\mathcal{C}^\perp$  is*

$$\mathcal{H}_S = \left( \left( \begin{array}{cc|ccc} T_b^t & I_{\alpha-\kappa} & \mathbf{0} & \mathbf{0} & 2S_b^t \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2I_{\gamma-\kappa} & 2R^t \\ T_2^t & \mathbf{0} & I_{\beta+\kappa-\gamma-\delta} & T_1^t & -(S_q + RT_1)^t \end{array} \right) \right), \quad (10)$$

where  $T_b, T_1, T_2, R, S_b$  are matrices over  $\mathbb{Z}_2$  and  $S_q$  is a matrix over  $\mathbb{Z}_4$ .

*Proof:* By Lemma 4, if  $\bar{\mathcal{C}}$  is a quaternary linear code with generator matrix

$$\bar{\mathcal{G}} = \left( \begin{array}{ccccc} 2T_b & 2T_2 & 2I_\kappa & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2T_1 & \mathbf{0} & 2I_{\gamma-\kappa} & \mathbf{0} \\ \hline 2S_b & S_q & \mathbf{0} & R & I_\delta \end{array} \right),$$

then the quaternary dual code  $\bar{\mathcal{C}}^\perp$  has generator matrix  $\bar{\mathcal{H}} =$

$$\left( \begin{array}{ccccc} \mathbf{0} & \mathbf{0} & 2I_\kappa & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2I_{\gamma-\kappa} & 2R^t \\ \hline I_{\alpha-\kappa} & \mathbf{0} & T_b^t & \mathbf{0} & 2S_b^t \\ \mathbf{0} & I_{\beta-\gamma-\delta+\kappa} & T_2^t & T_1^t & -(S_q + RT_1)^t \end{array} \right).$$

Hence, if  $\mathcal{C}$  is a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code with generator matrix (3), then the generator matrix of  $\chi(\mathcal{C})^\perp$  is  $\mathcal{H}_\xi =$

$$\left( \begin{array}{ccccc} 2I_\kappa & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2I_{\gamma-\kappa} & 2R^t \\ \hline T_b^t & I_{\alpha-\kappa} & \mathbf{0} & \mathbf{0} & 2S_b^t \\ T_2^t & \mathbf{0} & I_{\beta-\gamma-\delta+\kappa} & T_1^t & -(S_q + RT_1)^t \end{array} \right).$$

Finally, by Proposition 2,  $\mathcal{H}_\mathcal{S} = \xi(\mathcal{H}_\xi)$  is the generator matrix of  $\mathcal{C}^\perp$ .  $\triangle$

Note that by Theorem 2 and Theorem 3, if  $\mathcal{C}$  is a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$  with canonical generator matrix (3), then  $\mathcal{C}^\perp$  is permutation equivalent to a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code with canonical generator matrix

$$\left( \begin{array}{cc|ccc} I_{\bar{\kappa}} & T_b^t & 2S_b^t & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2R^t & 2I_{\bar{\gamma}-\bar{\kappa}} & \mathbf{0} \\ \hline \mathbf{0} & T_2^t & -(S_q + RT_1)^t & T_1^t & I_{\bar{\delta}} \end{array} \right), \quad (11)$$

where  $T_b, T_1, T_2, R, S_b$  are matrices over  $\mathbb{Z}_2$ ;  $S_q$  is a matrix over  $\mathbb{Z}_4$ ,  $\bar{\gamma} = \alpha + \gamma - 2\kappa$ ,  $\bar{\delta} = \beta - \gamma - \delta + \kappa$  and  $\bar{\kappa} = \alpha - \kappa$ .

**Example 5** Let  $\mathcal{C}_{S1}$  denote the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of type  $(1, 3; 1, 2; 1)$  with canonical generator matrix

$$\mathcal{G}_S = \left( \begin{array}{c|ccc} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array} \right).$$



By Theorem 2, the additive dual code  $\mathcal{C}_{S1}^\perp$  is of type  $(1, 3; 0, 1; 0)$ . There are two methods to obtain a parity-check matrix of  $\mathcal{C}_{S1}$  from the matrix  $\mathcal{G}_S$ .

The first one uses Proposition 2. We know that if  $\bar{\mathcal{C}}$  is a quaternary linear code with generator matrix  $\bar{\mathcal{G}} = \left( \begin{array}{cccc} 2 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{array} \right)$ , the quaternary dual code  $\bar{\mathcal{C}}^\perp$  has generator matrix  $\bar{\mathcal{H}} = \left( \begin{array}{cccc} 0 & 2 & 0 & 0 \\ 1 & 1 & 3 & 1 \end{array} \right)$ . So, the generator matrix of  $\chi(\mathcal{C}_{S1})^\perp$  is  $\left( \begin{array}{cccc} 2 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 \end{array} \right)$  and finally, applying  $\xi$ , the generator matrix of  $\mathcal{C}_{S1}^\perp = \xi(\chi(\mathcal{C}_{S1})^\perp)$  is

$$\mathcal{H}_S = ( \begin{array}{c|ccc} 1 & 1 & 3 & 1 \end{array} ).$$

The second method uses Proposition 3. We know that the quaternary linear code  $\xi^{-1}(\mathcal{C}_{S1})$  with generator matrix

$$\left( \begin{array}{cccc} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array} \right),$$

or equivalently  $\left( \begin{array}{cccc} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array} \right)$ , has parity-check matrix  $\left( \begin{array}{cccc} 2 & 1 & 3 & 1 \end{array} \right)$ . So, applying  $\chi^{-1}$ , the generator matrix of  $\mathcal{C}_{S1}^\perp = \chi^{-1}(\xi^{-1}(\mathcal{C}_{S1})^\perp)$  is

$$\mathcal{H}_S = ( \begin{array}{c|ccc} 1 & 1 & 3 & 1 \end{array} ).$$

**Example 6** Let  $\mathcal{C}_{S2}$  be a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of type  $(3, 4; 3, 1; 1)$  with canonical generator matrix

$$\left( \begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right).$$

By Theorem 2 and Theorem 3, the additive dual code  $\mathcal{C}_{S2}^\perp$  is of type  $(3, 4; 0, 3; 0)$  and has generator matrix

$$\left( \begin{array}{ccc|cccc} 1 & 0 & 1 & 1 & 0 & 0 & 3 \\ 1 & 0 & 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{array} \right).$$

## 6 Additive self-dual codes

Let  $\mathcal{C}$  be a  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code. We say that  $\mathcal{C}$  is an *additive self-orthogonal code* if  $\mathcal{C} \subseteq \mathcal{C}^\perp$  and  $\mathcal{C}$  is an *additive self-dual code* if  $\mathcal{C} = \mathcal{C}^\perp$ . Let  $C = \Phi(\mathcal{C})$  be the corresponding  $\mathbb{Z}_2\mathbb{Z}_4$ -linear code. We say that  $C$  is a *self  $\mathbb{Z}_2\mathbb{Z}_4$ -orthogonal code* if  $C \subseteq C_\perp$  and  $C$  is a *self  $\mathbb{Z}_2\mathbb{Z}_4$ -dual code* if  $C = C_\perp$ , where  $C_\perp = \Phi(\mathcal{C}^\perp)$ . In this section, we will study the additive self-dual codes.

Note that in the case that  $\beta = 0$ , that is when  $\mathcal{C} = C$  is a binary linear code, we will also say that  $\mathcal{C}$  is *binary self-orthogonal* (or *binary self-dual*) if  $\mathcal{C} \subseteq \mathcal{C}^\perp$  (or  $\mathcal{C} = \mathcal{C}^\perp$ ). And in the case that  $\alpha = 0$ , that is when  $\mathcal{C}$  is a quaternary linear code, we will also say that  $\mathcal{C}$  is *quaternary self-orthogonal* (or *quaternary self-dual*) if  $\mathcal{C} \subseteq \mathcal{C}^\perp$  (or  $\mathcal{C} = \mathcal{C}^\perp$ ).

Recall that  $\mathcal{C}_X$  is the punctured code of  $\mathcal{C}$  by deleting the coordinates outside  $X$ ,  $\mathcal{C}_Y$  is the punctured code of  $\mathcal{C}$  by deleting the coordinates outside  $Y$  and  $\mathcal{C}_b$  is the subcode of  $\mathcal{C}$  which contains all codewords of order two. Denote by  $w(u)$  the Hamming weight of any vector  $u \in \mathbb{Z}_2^\alpha$ .

**Lemma 5** *If  $\mathcal{C}$  is an additive self-dual code, then  $\mathcal{C}$  is of type  $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$ ,  $|\mathcal{C}| = 2^{\kappa+\beta}$  and  $|\mathcal{C}_b| = 2^{\kappa+\beta-\delta}$ .*

*Proof:* By Theorem 2, we have that  $\alpha = 2\kappa$  and  $\gamma = \beta + \kappa - 2\delta$ . Since  $|\mathcal{C}| = 2^{\gamma+2\delta}$  and  $|\mathcal{C}_b| = 2^{\gamma+\delta}$ , the result holds.  $\triangle$

**Lemma 6** *Let  $\mathcal{C}$  be an additive self-dual code and let  $z = (x \mid y) \in \mathcal{C}$ . Denote by  $p(u)$  the number of odd (order four) coordinates of any vector  $u \in \mathbb{Z}_4^\beta$ . Then,*

- (i) *if  $w(x)$  is even, then  $p(y) \equiv 0 \pmod{4}$ .*
- (ii) *if  $w(x)$  is odd, then  $p(y) \equiv 2 \pmod{4}$ .*
- (iii)  *$(\mathbf{0} \mid \mathbf{2})$  is a codeword in  $\mathcal{C}$ .*

*Proof:* (i) and (ii) follows easily since  $z$  must be orthogonal to itself and we have  $\langle z, z \rangle = 2w(x) + p(y) = 0 \in \mathbb{Z}_4$ . Now, (iii) is obvious because  $p(y)$  is always even.  $\triangle$

**Lemma 7** *If  $\mathcal{C}$  is an additive self-dual code, then the subcode  $(\mathcal{C}_b)_X$  is a binary self-dual code.*

*Proof:* By Lemma 5, the code  $\mathcal{C}$  is of type  $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$ . Since for any pair of codewords  $(x \mid y), (x' \mid y') \in \mathcal{C}_b$  we have  $\langle y, y' \rangle_4 = 0$ ,  $(\mathcal{C}_b)_X \subseteq (\mathcal{C}_b)_X^\perp$ . Moreover, since  $(\mathcal{C}_b)_X$  has dimension  $\kappa$  (by definition) and is of length  $2\kappa$ , we have that  $(\mathcal{C}_b)_X$  is binary self-dual.  $\triangle$

**Lemma 8** *Let  $\mathcal{C}$  be an additive self-dual code of type  $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$ . There is an integer number  $r$ ,  $0 \leq r \leq \kappa$ , such that each codeword in  $\mathcal{C}_Y$  appears  $2^r$  times in  $\mathcal{C}$  and  $|\mathcal{C}_Y| \geq 2^\beta$ .*

*Proof:* Consider the subcode  $\mathcal{C}_0 = \{(x \mid \mathbf{0}) \in \mathcal{C}\}$ . Clearly,  $(\mathcal{C}_0)_X$  is a binary linear code. Let  $r = \dim(\mathcal{C}_0)_X$ . Thus, any vector in  $\mathcal{C}_Y$  appears  $2^r$  times in  $\mathcal{C}$ . Note that  $(\mathcal{C}_0)_X$  is also a subcode of  $(\mathcal{C}_b)_X$ , hence  $r \leq \kappa$ . Also, we have that  $|\mathcal{C}| = 2^{\beta+\kappa} = |\mathcal{C}_Y| \cdot 2^r$ , therefore  $|\mathcal{C}_Y| \geq 2^\beta$ .  $\triangle$

We say that a binary code  $C$  is *antipodal* if for any codeword  $z \in C$ ,  $z + \mathbf{1} \in C$ . The following two examples show us two different cases of additive self-dual codes. In Example 7, the corresponding  $\mathbb{Z}_2\mathbb{Z}_4$ -linear code  $\mathcal{C}_1 = \Phi(\mathcal{C}_1)$  is antipodal, or equivalently  $\mathcal{C}_1$  contains the codeword  $(\mathbf{1} \mid \mathbf{2})$ . On the other hand, in Example 8, the corresponding  $\mathbb{Z}_2\mathbb{Z}_4$ -linear code  $\mathcal{C}_2 = \Phi(\mathcal{C}_2)$  is not antipodal. We will study these two cases separately.

**Example 7** *An additive self-dual code with  $\alpha, \beta \geq 1$  should have  $\alpha \geq 2$ , since  $\alpha$  must be even. An additive self-dual code with minimum number of coordinates has  $\alpha = 2$ ,  $\beta = 1$  and  $2^{\kappa+\beta} = 2^{1+1} = 4$  codewords. For example, the code  $\mathcal{C}_1 = \{(00 \mid 0), (00 \mid 2), (11 \mid 0), (11 \mid 2)\}$  is an additive self-dual code of type  $(2, 1; 2, 0; 1)$  and has generator matrix*

$$\mathcal{G}_1 = \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 2 \end{array} \right).$$

**Example 8** *Let  $\mathcal{C}_b = \{(00 \mid 00), (00 \mid 22), (11 \mid 02), (11 \mid 20)\}$ . Then, the code  $\mathcal{C}_2 = \mathcal{C}_b \cup (\mathcal{C}_b + (01 \mid 11))$  is an additive self-dual code of type  $(2, 2; 1, 1; 1)$  and has generator matrix*

$$\mathcal{G}_2 = \left( \begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{array} \right).$$

The following Lemmas 9 and 10 give us two generalizations of Example 8. Note that any of the corresponding  $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes are not antipodal.

**Lemma 9** *If  $\delta \leq \kappa$ , the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code  $\mathcal{C}$  of type  $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$  with canonical generator matrix*

$$\mathcal{G} = \left( \begin{array}{cccc|ccc} I_\delta & \mathbf{0} & I_\delta & \mathbf{0} & 2I_\delta & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{\kappa-\delta} & \mathbf{0} & I_{\kappa-\delta} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2I_{\beta-2\delta} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & I_\delta & \mathbf{0} & I_\delta & \mathbf{0} & I_\delta \end{array} \right)$$

*is an additive self-dual code.*

*Proof:* Straightforward using Theorem 3.  $\triangle$

**Lemma 10** *If  $\delta \leq \kappa$ , the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code  $\mathcal{C}$  of type  $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$  with canonical generator matrix*

$$\mathcal{G} = \left( \begin{array}{cccc|ccc} I_\delta & \mathbf{0} & I_\delta & \mathbf{0} & 2I_\delta & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{\kappa-\delta} & \mathbf{0} & I_{\kappa-\delta} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2} & 2I_{\beta-2\delta} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & I_\delta & \mathbf{0} & I_\delta & \mathbf{1} & I_\delta \end{array} \right)$$

*is an additive self-dual code if and only if  $\beta - 2\delta \equiv 0 \pmod{4}$ .*

*Proof:* Let  $e_i$  denote the vector with all components equal to zero, except the  $i$ th component, which contains a one. It is easy to see that any two rows of the generator matrix  $\mathcal{G}$  are orthogonal. Notice that the rows of order four  $u = (\mathbf{0} \ \mathbf{0} \ e_i \ \mathbf{0} \mid e_i \ \mathbf{1} \ e_i)$  are orthogonal if and only if  $\beta - 2\kappa \equiv 0 \pmod{4}$ . By Theorem 3, we have  $|\mathcal{C}| = |\mathcal{C}^\perp|$ , so  $\mathcal{C}$  is additive self-dual.  $\triangle$

**Proposition 4** *Let  $C$  be a self  $\mathbb{Z}_2\mathbb{Z}_4$ -dual code of length  $n$  and let  $A_i$  denote the number of codewords of weight  $i$  ( $0 \leq i \leq n$ ). The following statements are equivalent:*

- (i)  $C$  is antipodal.
- (ii)  $C_X$  has only even weights (and also  $C$ ).
- (iii)  $\sum_{i=0}^n (-1)^i A_i = |C|$ .

*Proof:* Let  $\mathcal{C}$  be the corresponding  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code  $\mathcal{C} = \Phi^{-1}(C)$ . Note that given a codeword  $z = (x \mid y) \in \mathcal{C}$ , where  $y = (y_1, \dots, y_\beta)$ , the weight of  $(\phi(y_1), \dots, \phi(y_\beta))$  is always even. Thus, the parity of the weight of  $\Phi(z)$  is the same as the parity of the weight of  $x$ .

(i)  $\Leftrightarrow$  (ii): If  $C$  is antipodal, then  $(\mathbf{1} \mid \mathbf{2}) \in \mathcal{C}$ . Therefore, for any codeword  $(x \mid y) \in \mathcal{C}$ ,  $w(x)$  must be even. Reciprocally, if  $w(x)$  is even for any codeword  $(x \mid y) \in \mathcal{C}$ , then the vector  $(\mathbf{1} \mid \mathbf{2})$  is orthogonal to any codeword and hence  $(\mathbf{1} \mid \mathbf{2}) \in \mathcal{C}$ .

(ii)  $\Leftrightarrow$  (iii): Straightforward because  $\sum_{i=0}^n A_i = |C|$ .  $\triangle$

**Proposition 5** *Let  $C$  be a self  $\mathbb{Z}_2\mathbb{Z}_4$ -dual code of length  $n$  and let  $A_i$  denote the number of codewords of weight  $i$  ( $0 \leq i \leq n$ ). The following statements are equivalent:*

(i)  $C$  is not antipodal.

(ii)  $C_X$  has even and odd weights (and also  $C$ ).

(iii)  $\sum_{i=0}^n (-1)^i A_i = 0$ .

*Proof:* Statements (i) and (ii) are equivalent by Proposition 4.

(i)  $\Leftrightarrow$  (iii): Consider the MacWilliams Identity:

$$W_C(X, Y) = \frac{1}{|C|} W_{C_\perp}(X + Y, X - Y),$$

where  $W_C(X, Y)$  is the weight enumerator polynomial of  $C$ :

$$W_C(X, Y) = \sum_{i=0}^n A_i X^{n-i} Y^i.$$

Since  $C = C_\perp$  and taking  $X = 0$ , we obtain:

$$A_n Y^n = \frac{1}{|C|} \sum_{i=0}^n (-1)^i A_i Y^n \implies |C| A_n = \sum_{i=0}^n (-1)^i A_i.$$

Finally, since  $A_n = 1$  when  $C$  is antipodal and  $A_n = 0$  when  $C$  is not antipodal, we have that  $\sum_{i=0}^n (-1)^i A_i = 0$  if and only if  $C$  is not antipodal.

$\triangle$

**Proposition 6** *Let  $\mathcal{C}$  be an additive self-dual code of type  $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$ . The following statements are equivalent:*

(i)  $\mathcal{C}_X$  is binary self-orthogonal.

(ii)  $\mathcal{C}_X$  is binary self-dual.

(iii)  $|\mathcal{C}_X| = 2^\kappa$ .

(iv)  $\mathcal{C}_Y$  is a quaternary self-orthogonal code.

(v)  $\mathcal{C}_Y$  is a quaternary self-dual code.

(vi)  $|\mathcal{C}_Y| = 2^\beta$ .

(vii)  $\mathcal{C} = \mathcal{C}_X \oplus \mathcal{C}_Y$ .

*Proof:* (i)  $\Leftrightarrow$  (ii): By Lemma 7,  $|\mathcal{C}_X| \geq 2^\kappa$ , thus (i) and (ii) are equivalent statements.

(ii)  $\Leftrightarrow$  (iii): Clearly, (ii) implies (iii) and (iii) implies  $\mathcal{C}_X = (\mathcal{C}_b)_X$  and  $\mathcal{C}_X$  is binary self-dual, by Lemma 7.

(ii)  $\Leftrightarrow$  (v): Straightforward.

(iv)  $\Leftrightarrow$  (v): By Lemma 8,  $|\mathcal{C}_Y| \geq 2^\beta$ , thus (iv) and (v) are equivalent statements.

(ii)  $\Leftrightarrow$  (vii): If  $\mathcal{C}_X$  is binary self-dual, then  $\mathcal{C}_Y$  is quaternary self-dual,  $|\mathcal{C}_X| = 2^\kappa$  and  $|\mathcal{C}_Y| = 2^\beta$ . Since  $\mathcal{C} = 2^{\kappa+\beta}$  we have that the set of codewords in  $\mathcal{C}$  is  $\mathcal{C}_X \times \mathcal{C}_Y$ . Reciprocally, if  $\mathcal{C} = \mathcal{C}_X \oplus \mathcal{C}_Y$ , then  $(x \mid \mathbf{0}) \in \mathcal{C}$  for any  $x \in \mathcal{C}_X$  and  $\mathcal{C}_X$  must be a binary self-dual code. Also,  $(\mathbf{0} \mid y) \in \mathcal{C}$  for any  $y \in \mathcal{C}_Y$  and  $\mathcal{C}_Y$  must be a quaternary self-dual code.

(v)  $\Rightarrow$  (vi): Trivial.

(vi)  $\Rightarrow$  (iii): By Lemma 8, each vector in  $\mathcal{C}_Y$  appears  $2^\kappa$  times in  $\mathcal{C}$ . Thus, for any vector  $x_b \in (\mathcal{C}_b)_X$ , the vector  $(x_b \mid \mathbf{0})$  is a codeword in  $\mathcal{C}_b$ . This means that given any codeword  $(x \mid y) \in \mathcal{C}$ , we have that  $\langle x, x_b \rangle = 0$ , for all  $x_b \in (\mathcal{C}_b)_X$ , since  $\langle (x \mid y), (x_b, \mathbf{0}) \rangle = 0$ . Therefore, for all  $x \in \mathcal{C}_X$ ,  $x \in (\mathcal{C}_b)_X^\perp$  and  $\mathcal{C}_X \subseteq (\mathcal{C}_b)_X^\perp$ . By Lemma 7,  $(\mathcal{C}_b)_X^\perp = (\mathcal{C}_b)_X$ , which implies  $\mathcal{C}_X = (\mathcal{C}_b)_X$ , hence  $|\mathcal{C}_X| = 2^\kappa$ .  $\triangle$

It is easy to check that if  $\mathcal{C}$  is an additive self-dual code, then the codewords in  $\mathcal{C}_X^\perp \oplus \mathcal{C}_Y^\perp$  are orthogonal to  $\mathcal{C}$  and, hence,  $\mathcal{C}_X^\perp \oplus \mathcal{C}_Y^\perp \subseteq \mathcal{C}^\perp$ .

**Proposition 7** *If  $\mathcal{C}_X$  is a binary self-dual code of length  $\alpha = 2\kappa$  and  $\mathcal{C}_Y$  is a quaternary self-dual code of type  $(0, \beta; \gamma, \delta; 0)$ , then  $\mathcal{C} = \mathcal{C}_X \oplus \mathcal{C}_Y$  is an additive self-dual code of type  $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$ .*

*Proof:* Since  $\mathcal{C}_X$  is binary self-dual,  $|\mathcal{C}_X| = 2^\kappa$ , where  $\alpha = 2\kappa$ . Since  $\mathcal{C}_Y$  is quaternary self-dual,  $|\mathcal{C}_Y| = 2^\beta = 2^{\gamma+2\delta}$ , so  $\gamma = \beta - 2\delta$ . Let  $\mathcal{G}_X$  be a generator matrix of  $\mathcal{C}_X$  of size  $\kappa \times 2\kappa$  and let  $\mathcal{G}_Y$  be a generator matrix of  $\mathcal{C}_Y$  of size  $(\beta - \delta) \times \beta$ . Then, the matrix  $\mathcal{G}$  defined as

$$\mathcal{G} = \left( \begin{array}{c|c} \mathcal{G}_X & \mathbf{0} \\ \hline \mathbf{0} & \mathcal{G}_Y \end{array} \right)$$

is a generator matrix of  $\mathcal{C} = \mathcal{C}_X \oplus \mathcal{C}_Y$ . It is easy to check that  $\mathcal{C}$  is an additive self-dual code of type  $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$  and  $|\mathcal{C}| = 2^{\kappa+\beta}$ .  $\triangle$

Clearly, any of the statements (i) – (vii) of Proposition 6 implies that  $C = \Phi(\mathcal{C})$  is antipodal. We are going to see that the converse is not true.

**Lemma 11** *Let  $\mathcal{C} \subset \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$  ( $\alpha, \beta \geq 1$ ) be an additive self-dual code such that  $C = \Phi(\mathcal{C})$  is antipodal (i.e.  $(\mathbf{1} \mid \mathbf{2}) \in \mathcal{C}$ ) and  $\mathcal{C}_X$  is not self-dual, then  $\alpha \geq 4$  and  $\beta \geq 4$ .*

*Proof:* Recall that  $\alpha$  must be even for an additive self-dual code. We are assuming that  $(\mathbf{1} \mid \mathbf{2}) \in \mathcal{C}$  and by Lemma 6  $(\mathbf{0} \mid \mathbf{2}) \in \mathcal{C}$ , thus  $(\mathbf{1} \mid \mathbf{0}) \in \mathcal{C}$  implying that any codeword has even weight in its binary coordinates.

If  $\alpha = 2$ , then  $\mathcal{C}_X = \{(0, 0), (1, 1)\}$ , which is self-dual. Therefore  $\alpha \geq 4$ .

Let  $x = (x_b \mid x_q)$  and  $y = (y_b \mid y_q)$  be two codewords such that  $x_b$  and  $y_b$  are not orthogonal. Then  $x_q$  and/or  $y_q$  must have order 4; otherwise  $x$  and  $y$  would not be orthogonal. Assume that  $x_q$  has order 4. Since  $w(x_b)$  is even, then  $x_q$  has at least 4 coordinates of order 4, by Lemma 6. Hence  $\beta \geq 4$ .  $\triangle$

If we assume that such an additive self-dual code  $\mathcal{C}$  of type  $(4, 4; \gamma, \delta; 2)$  exists, the sum of two codewords of order 4 always gives an order 2 codeword. Hence it will have the same number of order 2 and order 4 codewords. Thus

$$2^{\gamma+\delta} = \frac{1}{2} \cdot 2^{\gamma+2\delta}$$

and we obtain that  $\delta = 1$ . By Lemma 5, we have  $\gamma + 2\delta = \beta + \kappa = 4 + 2 = 6$ , which implies that  $\gamma = 4$ . Effectively such a code  $\mathcal{C}$  exists. A generator matrix for  $\mathcal{C}$  is

$$\left( \begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right).$$

Therefore, we have proven the following result:

**Proposition 8** *An additive self-dual code  $\mathcal{C}$  with minimum cardinality and number of coordinates such that  $C = \Phi(\mathcal{C})$  is antipodal and  $\mathcal{C}_X$  is not self-dual is of type  $(4, 4; 4, 1; 2)$ .*

The following lemma give us a family of additive self-dual codes  $\mathcal{C}$  such that  $C = \Phi(\mathcal{C})$  is antipodal and  $\mathcal{C}_X$  is not self-dual.

**Lemma 12** *If  $\delta < \kappa$ , the  $\mathbb{Z}_2\mathbb{Z}_4$ -additive code  $\mathcal{C}$  of type  $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$  with canonical generator matrix*

$$\mathcal{G} = \left( \begin{array}{cccccc|ccc} I_\delta & \mathbf{0} & \mathbf{0} & I_\delta & \mathbf{0} & \mathbf{0} & 2I_\delta & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{\kappa-\delta-1} & \mathbf{0} & \mathbf{0} & I_{\kappa-\delta-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2} & 2I_{\beta-2\delta} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & I_\delta & 1 & \mathbf{0} & I_\delta & 1 & I_\delta \end{array} \right)$$

*is an additive self-dual code if and only if  $\beta - 2\delta \equiv 2 \pmod{4}$ . Moreover,  $C = \Phi(\mathcal{C})$  is antipodal and  $\mathcal{C}_X$  is not self-dual.*

*Proof:* Using the same arguments as in the proof of Lemma 10, we have that  $\mathcal{C}$  is an additive self-dual code if and only if  $\beta - 2\delta \equiv 2 \pmod{4}$ . By Proposition 4,  $C = \Phi(\mathcal{C})$  is antipodal. And, it is clear that  $\mathcal{C}_X$  is not self-dual.  $\triangle$

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